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On a Class of Exact Geodesics of the Erez–Rosen Metric

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Abstract

In a previous paper, a class of exact geodesics for the motion of a particle in a gravitational-monopole-prolate-quadrupole field was investigated, both in Newtonian mechanics and in general relativity. This paper consists of both an amplification of the analysis contained in the previous paper and an extension of the analysis to the case for negative quadrupole moment, which was not treated previously. The relativistic results are based on the monopole-quadrupole metric of Erez and Rosen, the Newtonian results on the monopole-quadrupole potential of Laplace. In the limit of vanishing quadrupole parameter ($q \to 0$), the relativistic results reduce to those of the familiar Schwarzschild case; in the weak-field limit ($r/m \to \infty$), the relativistic results reduce to those of the Newtonian case. The existence and stability thresholds in the relativistic case yield values that uniquely characterize the Erez–Rosen metric.

1. *Introduction*

In a previous paper (Armenti and Havas, 1971; referred to hereafter as Paper I) a class of exact solutions for the motion of a particle in a gravitational-monopole-prolate-quadrupole field was investigated, both in Newtonian mechanics and general relativity. This class of exact solutions consisted partly of "circular noncoplanar motions," i.e., circular motions with constant angular velocity in planes parallel to the plane of symmetry of the quadrupole; the remainder consisted of circular motions with constant angular velocity *in* the plane of symmetry of the quadrupole. The circular noncoplanar motions exist only for positive quadrupole moments, i.e., only for *prolate* quadrupole fields. Circular motions in the plane of symmetry exist both for *negative* as well as positive quadrupole moments, although the former case was not treated in Paper I (for an extended body, negative quadrupole moment corresponds to an *oblate* distribution of mass).

This paper consists both of an amplification of certain results for positive quadrupole moment, presented in Paper I, and an extension of the analysis to the case for negative quadrupole moment.

The calculations of this paper are based on an exact solution of Einstein's vacuum field equations, the monopole-quadrupole metric of Erez and Rosen (1959). In Schwarzschild coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, with units such that $c = 1$, it equals

$$ds^2 = e^{2\psi} dt^2 - e^{2\gamma - 2\psi} \left[\left(1 + \frac{m^2 \sin^2\theta}{r^2 - 2mr}\right) dr^2 + (r^2 - 2mr + m^2 \sin^2\theta) d\theta^2 \right]$$
$$- e^{-2\psi}(r^2 - 2mr) \sin^2\theta d\phi^2 \tag{1.1}$$

where ψ and γ are complicated functions depending on both r and θ and involving both a mass and a quadrupole parameter, m and q. These are related to the familiar Newtonian parameters M and Q by $m = GM$ and $q = 15GQ/2m^3$.

2. *Existence of Circular Noncoplanar Motions*

As shown in Paper I, the relativistic analogs of energy and angular momentum E and C, for a test particle executing circular noncoplanar motion in the gravitational field (1.1) are given by

$$E^2 = \frac{[(r - m)(g + 3h) - g(m + 2qh)] e^{2\psi}}{(r - m)(g + 3h) - 2g(m + 2qh)} \tag{2.1}$$

and

$$C^2 = \frac{g(r^2 - 2mr)(m + 2qh)^2 e^{-2\psi}}{q(g + 3h)[(r - m)(g + 3h) - 2g(m + 2qh)]} \tag{2.2}$$

where

$$g \equiv -\tfrac{3}{2}(r - m) \left[\frac{1}{2}\left(\frac{3r^2}{m^2} - \frac{6r}{m} + 2\right) \ln\left(1 - \frac{2m}{r}\right) + \frac{3}{m}(r - m) \right] \tag{2.3}$$

and

$$h \equiv \tfrac{1}{2}(r^2 - 2mr) \left[\frac{3}{2m^2} (r - m) \ln\left(1 - \frac{2m}{r}\right) + \frac{m}{2(r^2 - 2mr)}\left(\frac{3r^2}{m^2} - \frac{6r}{m} + 2\right) + \frac{3}{2m} \right] \tag{2.4}$$

In order for both E and C to be real and finite (in view of the fact that q must here be positive), it is necessary that

$$(r - m)(g + 3h) - 2g(m + 2qh) > 0 \tag{2.5}$$

From this it is apparent that the allowed values of q are bounded from above by a value

$$q_a \equiv \frac{1}{2h}\left[\frac{(r - m)(g + 3h)}{2g} - m\right] \tag{2.6}$$

In addition, true noncoplanar motions require that

$$\sin^2\theta < 1 \tag{2.7}$$

Now from equation (33) of Paper I, a necessary condition for the existence of circular noncoplanar motions in the field (1.1) is

$$\sin^2\theta = S(r, q) \equiv \frac{m/q + 2h}{g + 3h} \tag{2.8}$$

From this condition and equation (2.7) one finds that q is also bounded from below, by a value

$$q_b \equiv \frac{m}{g + h} \tag{2.9}$$

Hence, real circular noncoplanar motions exist only for those values of q that satisfy

$$q_b < q < q_a \tag{2.10}$$

In Figure 1 we plot the "existence region" for circular noncoplanar motions

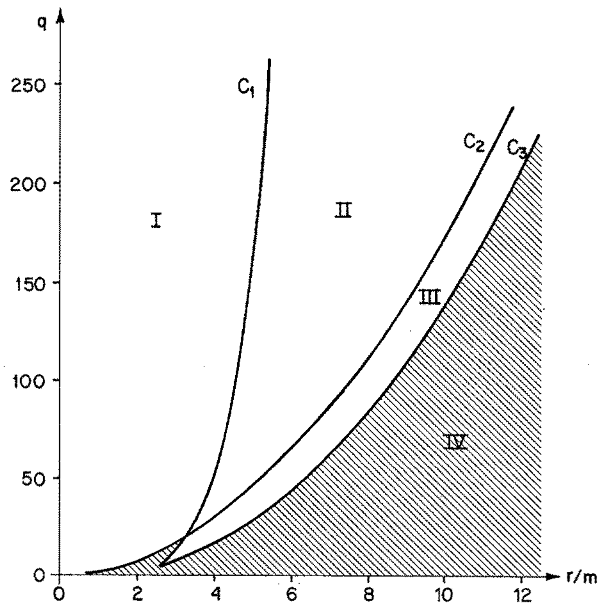

Figure 1. Existence region in $(q, r/m)$ plane for circular noncoplanar motion. $C_1: q = q_a$; $C_2: q = 5r^2/3m^2$; $C_3: q = q_b$. Newtonian circular noncoplanar motions are possible in regions I and II. Relativistic circular noncoplanar motions are possible only in regions II and III. Neither theory permits such motions in region IV. For the Newtonian case, circular noncoplanar motions exist down to $r = 0, q = 0$; in the relativistic case, they exist only down to $r = 2.4481m, q = 2.2544$.

in the Erez-Rosen monopole-quadrupole field, that is, the region of the $(q, r/m)$ plane defined by (2.10).

Also shown in Figure 1 and in certain other figures that follow are the results of the corresponding Newtonian calculations, which are based on the monopole-quadrupole potential

$$V = -\frac{GM}{r} + \frac{GQ}{2r^3}(1 - 3\cos^2\theta)$$

The Newtonian case is straightforward; the details may be found in Armenti (1970).

We see that there is an existence threshold value of q and a corresponding one for r below which circular noncoplanar motions do not exist, and whose values are determined by a simultaneous solution of equations (2.6) and (2.9). The results of this solution, found numerically, have the values (see Paper I) $q = q_{\text{et}} = 2.2544$ and $r = r_{\text{et}} = 2.4481m$, and uniquely characterize the Erez-Rosen solution.

For large values of r/m we can make use of equations (2.6) and (2.9), together with equations (37) and (38) of Paper I to obtain the approximate expressions

$$q_a = \frac{25}{8}\left(\frac{r}{m} - 1\right)^2 \left(\frac{r}{m} - \frac{9}{5}\right) \tag{2.11}$$

and

$$q_b = \frac{5}{3}\left(\frac{r}{m} - 1\right)^2 \tag{2.12}$$

From these equations it is clear that in the $(q, r/m)$ plane the existence region widens indefinitely with increasing r/m.

It is instructive to consider the condition on the azimuth following from condition (2.10). If we solve for q in equation (2.8) and substitute that expression into (2.10) we obtain

$$J(r) < \sin^2\theta < 1 \tag{2.13}$$

where

$$J(r) \equiv \frac{2h}{g + 3h}\left[1 + \frac{2gm}{(r - m)(g + 3h) - 2gm}\right] \tag{2.14}$$

In Figure 2 we plot the existence region in the $(\sin^2\theta, r/m)$ plane, i.e., the region defined by equation (2.13). Finally, in Figure 3 we plot the existence region in the (R, z) plane when R and z are "radial cylindrical coordinates" defined by $R = r\sin\theta$ and $z = r\cos\theta$.

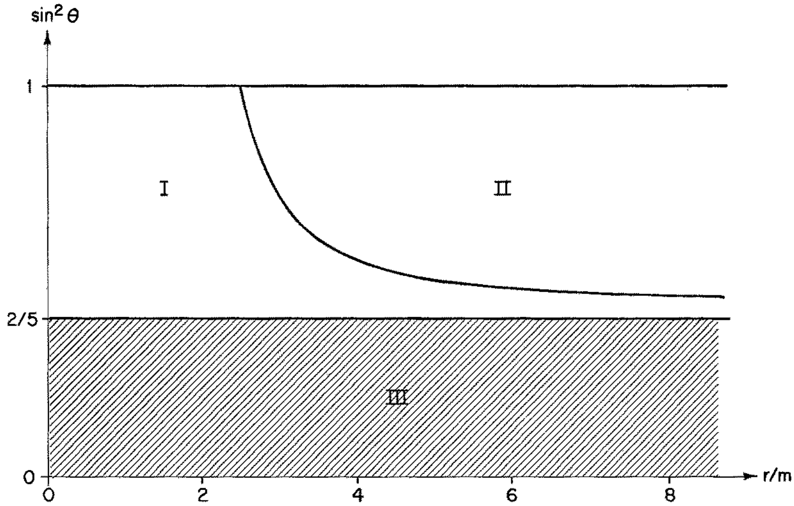

Figure 2. Existence region in $(\sin^2\theta, r/m)$ plane for circular noncoplanar motion. Newtonian circular noncoplanar motions are possible in regions I and II. Relativistic circular noncoplanar motions are possible only in region II. Neither theory permits such motions in region III. The relativistic existence condition is defined by equation (2.13). The Newtonian existence minimum occurs at $\sin^2\theta = 2/5$. The relativistic existence minimum approaches this value asymptotically as $(r/m) \to \infty$.

3. *Existence of Circular Motion in the Plane of Symmetry*

Circular motions exist in the plane of symmetry for *both* signs of the quadrupole moment. The energy and angular momentum for a test particle moving in a circle with constant angular velocity in such a plane is given by (see Paper I)

$$E^2 = \frac{(r - 2m + qh)e^{2\psi}}{r - 3m + 2qh} \tag{3.1}$$

and

$$C^2 = \frac{(m - qh)(r^2 - 2mr)e^{-2\psi}}{(r - 3m + 2qh)} \tag{3.2}$$

Again, physical motions require an energy and angular momentum that are both real and finite. Hence, the existence criteria for circular motion in the plane of symmetry (for both signs of q) become

$$r - 3m + 2qh > 0 \tag{3.3}$$

and

$$m - qh > 0 \tag{3.4}$$

These equations provide, respectively, a lower and an upper bound on the allowed values of q. In fact, the only circular motions that can exist in the

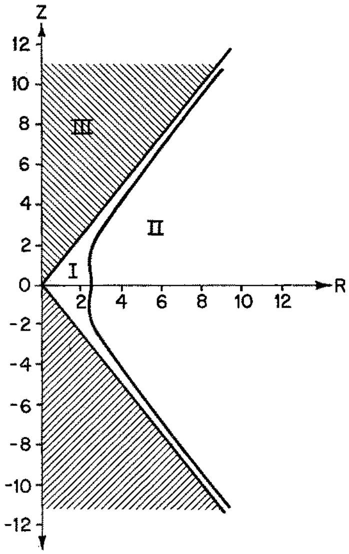

Figure 3. Existence region in (R, z) plane for circular noncoplanar motion. Newtonian circular noncoplanar motions are possible in regions I and II. Relativistic circular noncoplanar motions are possible only in region II. Neither theory permits such motions in region III. Note that in the relativistic case smaller circles are possible *outside* the plane of symmetry than within. In fact, the smallest circles occur for $z = \pm 1.20127m$, with a radius $R = 2.37704m$, and with $r = 2.66335m$, $\sin^2\theta = 0.796554$, and $q = 4.99815$.

plane of symmetry are those that satisfy

$$\frac{3m - r}{2h} < q < \frac{m}{h} \tag{3.5}$$

In Figure 4 we plot the existence region for circular motion of a test particle in the plane of symmetry of the Erez-Rosen monopole-quadrupole field. This is, the region defined by (3.5).

4. *Stability of Circular Noncoplanar Motions*

The stability of the motions whose existence was discussed in Section 2 is conveniently studied by means of the standard stability analysis based on the Lagrangian formalism (Whittaker, 1937). One finds a condition on the azimuth (Armenti, 1970)

$$\sin^2\theta > K(r, q) \tag{4.1}$$

where

$$K(r, q) \equiv \frac{a + b - c}{d} \tag{4.2}$$

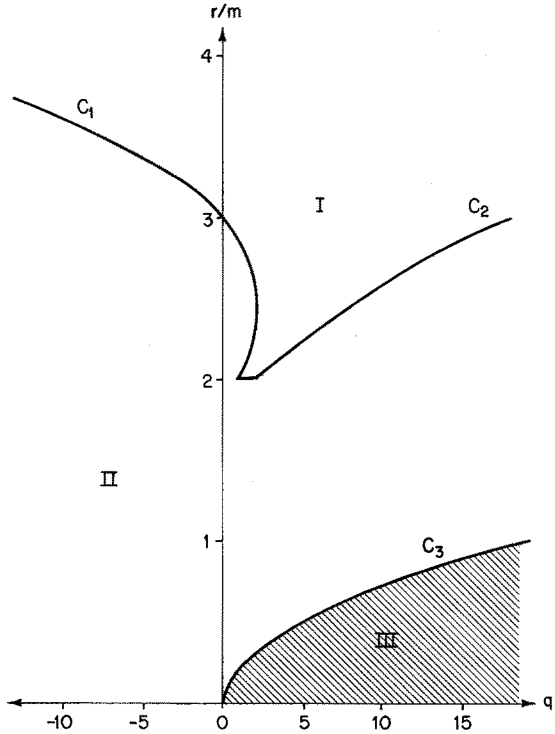

Figure 4. Existence region in $(q, r/m)$ plane for circular motion in the plane of symmetry. In the Newtonian case, circular motions are possible in regions I and II. For the relativistic case they exist only in region I. Neither theory permits such motions in region III. Newtonian circular motions are possible in the plane of symmetry for all $r > 0$ so long as $q < 0$ (*oblate* distribution of mass), but are possible only for $r > (3q/5m^2)^{1/2}$ for $q > 0$ (*prolate* distribution of mass). For the relativistic case, smaller circles are possible in the plane of symmetry for $q > 0$ than for $q < 0$; e.g., for $1 < q < 2$ circles with $r = 2m$ are possible. For $q = 0$, the smallest possible circle has a radius $r = 3m$, in agreement with the Schwarzschild result. C_1: $q = (3m - r)/2h$; C_2: $q = m/h$; C_3: $q = 5r^2/3m^2$.

and

$$\begin{aligned}
a &\equiv 2g^2(g+3h)^3(r^2-2mr)(r-m)^2 \\
b &\equiv qg^2(m/q+2h)^2\,(g+3h)\{g(r^2-2mr)[3(r-m)+2qg]\; - 3(g+6h)(r-m)^3\} \\
c &\equiv 2q^2g^3(m/q+2h)^3\,[g(r^2-2mr)+(g+6h)(r-m)^2] \\
d &\equiv (r-m)(g+3h)^3\{g^2(r^2-2mr)[2(r-m)+3qg]\; + (g^2+6hg-9h^2)(r-m)^3\}
\end{aligned} \tag{4.3}$$

It is clear then that, for a given q, only those circular noncoplanar motions with azimuth satisfying

$$K(r, q) < \sin^2 \theta < 1 \tag{4.4}$$

will be stable motions. In Figures 5a, 5b, 5c and 5d, we plot simultaneously equations (2.8) and (4.2) for four different values of q. Also shown in these plots is the existence region defined by equation (2.13). We see that as q

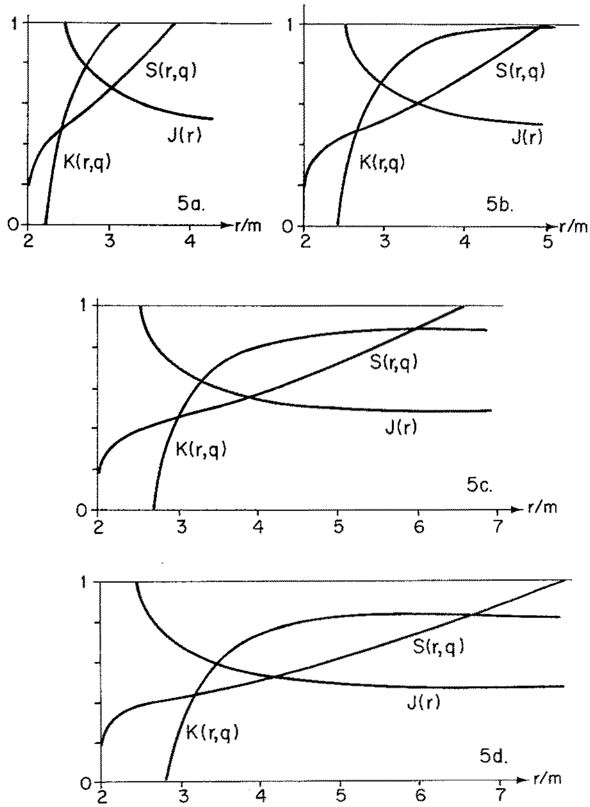

Figure 5. Relativistic stability minima for circular noncoplanar moton for various values of q. *Existence* minima occur at the point defined by $J(r) = S(r, q)$ [see equations (2.8) and (2.14)]. *Stability* minima occur at the intersection of $K(r, q)$ and $S(r, q)$ [see equation (4.2)]. 5a: For $q = 12$, there are no stable motions since equation (4.4) is not satisfied; 5b: For $q = 25$, the point of intersection of $K(r, q)$ and $S(r, q)$ falls just inside the allowed region and hence equation (4.4) is satisfied; 5c, 5d: For $q = 50$ and $q = 75$, the point of intersection falls well within the allowed region, and for these, as well as larger values of q, stable motions always exist.

increases, the minimum value of r for which stable motions occur continually increases, while the corresponding value of $\sin^2\theta$ continually decreases. For some values of q (Fig. 5a) there is no r_{s2} (defined below) whose corresponding azimuth falls in the allowed range (4.4), and hence such a value of q cannot lead to stable motion.

We note (Fig. 5b) that a "stability threshold" occurs near $q = 25$. To investigate this matter further we consider the locus of points defined by the intersection of equations (2.5) and (4.2) for positive values of q; that is we consider solutions of the transcendental equation

$$\frac{m/q + 2h}{g + 3h} = K(r, q) \tag{4.5}$$

As may be seen from Figure 5, equation (4.5) has *two* solutions in r for each value of q. We will denote these by $r_{s1}(q)$ and $r_{s2}(q)$, with $r_{s1} < r_{s2}$. It would appear at first glance that stable motions would then be possible in two distinct regions, namely, for $2m < r < r_{s1}$ and for $r_{s2} < r < r_{\max}$, with $r_{\max}(q)$ corresponding to $\sin^2\theta = 1$ (and being determined from $m/q = g + h$). It turns out, however (Figure 6), that for every value of q, $r_{s1}(q) < r_{em}(q)$, the existence minimum defined by equation (2.5). We see then that although the stability condition (4.4) is satisfied in the region $2m < r < r_{s1}$, circular noncoplanar solutions do not exist there. Hence, for each value of q we have at most one stable region, namely, that corresponding to

$$r_{s2}(q) < r < r_{\max}(q) \tag{4.6}$$

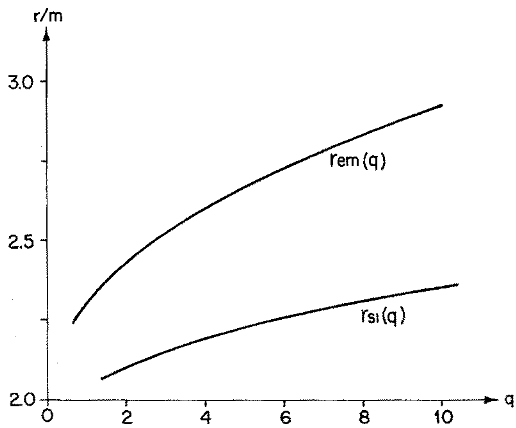

Figure 6. A weak stability condition for relativistic circular noncoplanar motion. The relativistic stability conditions allow for stable motions in the region $2m < r < r_{s1}(q)$, but the relativistic existence conditions demand that $r > r_{em}(q)$. Since $r_{s1}(q)$ is everywhere less than $r_{em}(q)$, this stability condition can never be satisfied and hence may be discarded.

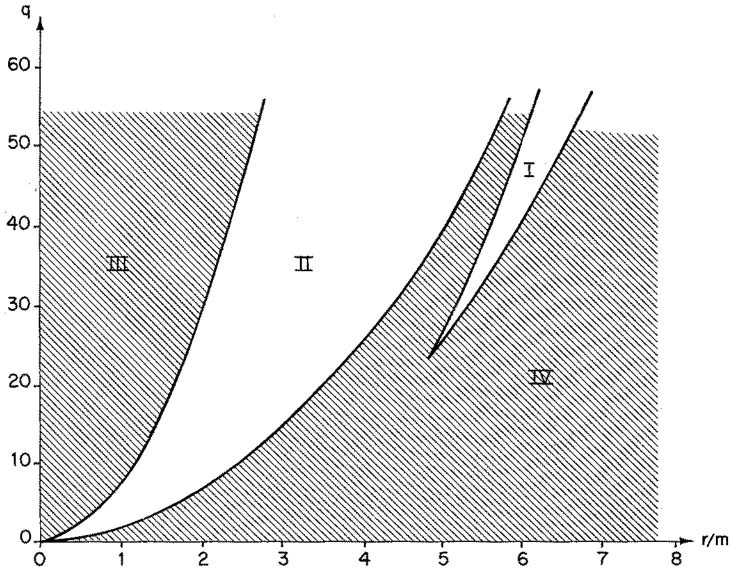

Figure 7. Stability region in $(q, r/m)$ plane for circular noncoplanar motion. In the relativistic case, stable circular noncoplanar motions are possible only in region I defined by $r_{s2}(q) < r < r_{max}(q)$, while in the Newtonian case, stable circular noncoplanar motons are possible only in region II defined by $(2q/15m^2)^{1/2} < r < (3q/5m^2)$. Regions I and II merge at the point $r = 6.8481m$, $q = 78.1599$. In the relativistic case, the condition $r_{s2}(q) = r_{max}(q)$ defines a stability threshold with coordinates $r_{st} = 4.9061m$ and $q_{st} = 24.2333$. In the Newtonian case, $r_{st} = q_{st} = 0$.

In Figure 7 we plot the stability region in the $(q, r/m)$ plane as defined by equation (4.6). We see that there is indeed a *stability threshold* and that the threshold values (found numerically) are $r_{st} = 4.9061m$ and $q_{st} = 24.2333$. No stable circular noncoplanar motions are possible below these values. The corresponding stability regions in the $(\sin^2\theta, r/m)$ and (R, z) planes are given in Figures 8 and 9.

5. *Stability of Circular Motions in the Plane of Symmetry*

For circular motions in the plane of symmetry, the standard stability analysis (Whittaker, 1937) leads to the two conditions (Armenti, 1970)

$$m - q(g + h) > 0 \tag{5.1}$$

and

$$\begin{aligned}\{2qh[9mqh + 8mr - 13m^2 - (r + qh)(r + 2qh)]\\ + 2m(r - 3m)(r - 2m) - (r^2 - 2mr)[m - q(g + h)]\} > 0\end{aligned} \tag{5.2}$$

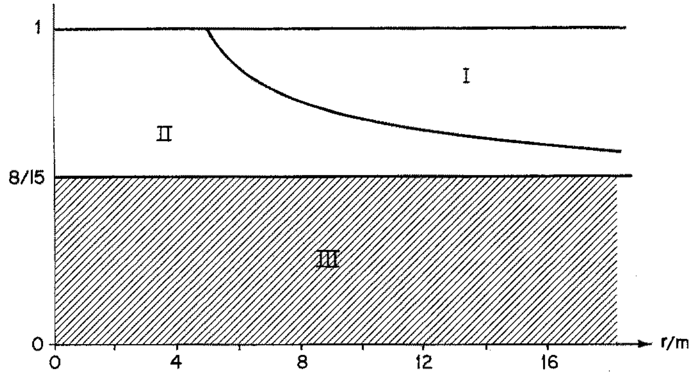

Figure 8. Stability region in $(\sin^2\theta, r/m)$ plane for circular noncoplanar motion. Stable relativistic motions are possible only in region I, while stable Newtonian motions are possible both in regions I and II. Neither theory allows for stable motions in region III. The Newtonian stability minimum occurs at $\sin^2\theta = 8/15$. The relativistic minimum approaches this value asymptotically as $(r/m) \to \infty$.

Viewed as inequalities on q, these equations take the forms

$$q - \frac{m}{g+h} < 0 \tag{5.3}$$

and

$$q^3 + a_2 q^2 + a_1 q + a_0 < 0 \tag{5.4}$$

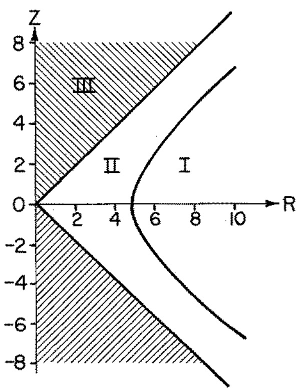

Figure 9. Stability region in (R, z) plane for circular noncoplanar motion. Stable relativistic motions are possible only in region I, while stable Newtoinan motions are possible in both regions I and II. Neither theory permits stable motions in region III.

where

$$a_2 \equiv \frac{3(r-3m)}{2h} \tag{5.5}$$

$$a_1 \equiv \frac{-[2h(8mr - 13m^2 - r^2) + (r^2 - 2mr)(g+h)]}{4h^3} \tag{5.6}$$

and

$$a_0 \equiv \frac{-m(r-2m)(r-6m)}{4h^3} \tag{5.7}$$

We will let $q_0(r)$, $q_1(r)$, and $q_2(r)$ denote the roots of the cubic expression (5.4) and will choose these labels in such a way that $q_0(r) < q_1(r) < q_2(r)$ at $r = 2m$. Then in order for equation (5.4) to be satisfied, q must satisfy either

$$q < q_0 \tag{5.8}$$

or

$$q_1 < q < q_2 \tag{5.9}$$

The nature of the three roots is determined by the sign of the quantity

$$\Delta \equiv s^3 + t^2 \tag{5.10}$$

where

$$s \equiv \tfrac{1}{3}a_1 - \tfrac{1}{9}a_2{}^2 \tag{5.11}$$

and

$$t \equiv \tfrac{1}{6}(a_1 a_2 - 3a_0) - \tfrac{1}{27}a_2{}^3 \tag{5.12}$$

For our case

$$s = \frac{-[(r-m)^2 h + (r^2 - 2mr)(g+h)]}{12h^3} \tag{5.13}$$

and

$$t = \frac{-(r^2 - 2mr)}{8h^3}\left[m + \frac{(r-3m)(g+h)}{2h}\right] \tag{5.14}$$

It follows from equations (5.10), (5.13), and (5.14) that $\Delta < 0$ for all $r > 2m$ and that therefore the roots of the cubic expression (5.4) are real and distinct for all $r > 2m$.

The values of the three roots are given explicitly by (Barrington, 1956)

$$q_k = (X_k - \tfrac{1}{3}a_2), \qquad k = 0, 1, 2 \tag{5.15}$$

with

$$X_k \equiv 2(-s)^{1/2} \cos\left[\tfrac{1}{3}\phi + (k+1)\tfrac{2}{3}\pi\right] \tag{5.16}$$

$$\cos\phi \equiv t/(-s^3)^{1/2} \tag{5.17}$$

and with s and t given by equations (5.13) and (5.14). At $r = 2m$, the roots $q_k = k$.

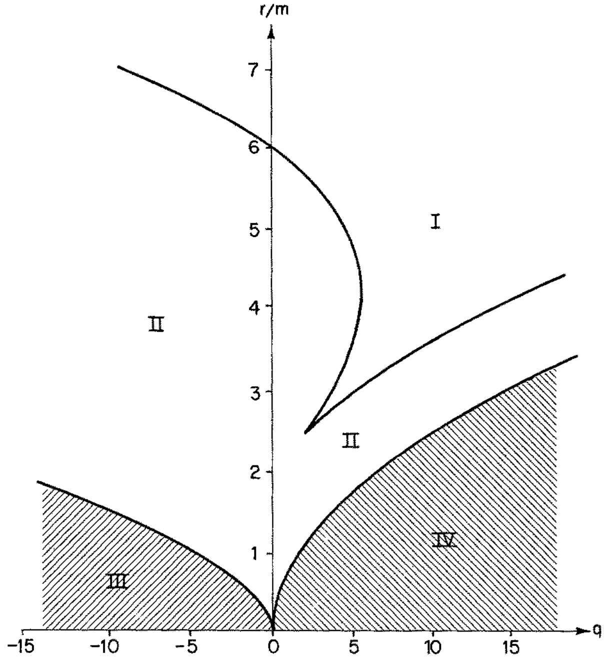

Figure 10. Stability region in $(q, r/m)$ plane for circular motion in the plane of symmetry. In the relativistic case, stable circular motions are possible only in region I defined by $q_1 < q < m/(g + h)$, for both signs of q. In the Newtonian case, stable circular motions are possible both in regions I and II; the precise conditions being $r > (-q/5m^2)^{1/2}$ for $q < 0$, and $r > (3q/5m^2)^{1/2}$ for $q > 0$. Neither theory allows for stable motions in the plane of symmetry in regions III and IV. In the relativistic case we find that for $q = 0$, the stability condition becomes $r > 6m$, in agreement with the Schwarzschild result.

The stability criteria for circular motions in the plane of symmetry are satisfied in the region defined by equations (5.3), (5.8), and (5.9). This region is shown in Figure 10. We note first that the region defined by equation (5.8) must be ruled out since circular motions do not exist for $q < q_0$. In addition, equations (5.3) and (5.9) together rule out the region $m/(g + h) < q < q_2$. Hence the stability region for circular motion in the plane of symmetry of an Erez-Rosen monopole-quadrupole field is defined by

$$q_1 < q < \frac{m}{g + h} \tag{5.18}$$

It is a remarkable fact that the curves $q = (3m - r)/2h$, $q = m/g + h)$, and $q = q_1$ all intersect at the same point, thereby defining a *stability threshold* for circular motion in the plane of symmetry. It is even more remarkable that the coordinates of this point are precisely the threshold values of r and q

obtained for the *existence* of circular noncoplanar motion in Section 2, viz., $r = r_{\text{et}}$ and $q = q_{\text{et}}$. To see how this comes about, we note that the r coordinate of the point of intersection of the curves $q = (3m - r)/2h$ and $q = m/(g + h)$ is determined by a solution of the equation

$$(3m - r)g = (r - m)h \tag{5.19}$$

When this equation is satisfied, however, we find from equations (5.14)-(5.17) that $t = x_1 = 0$ and that therefore $q_1 = (3m - r)/2h$. Hence the three curves do indeed intersect at the same point.

To see that the coordinates of the point of intersection are in fact $r = r_{\text{et}}$ and $q = q_{\text{et}}$, we recall that the existence threshold values of r and q were defined by a simultaneous solution of equations (2.6) and (2.9). After equating these two expressions and rearranging the result, we immediately regain equation (5.19). Thus we see that for *positive* q, stable circular motions in the plane of symmetry occur only for $r > r_{\text{et}}$. For $q = 0$, we find from (5.1) and (5.2) (and see from Figure 10) that $r > 6m$, the familiar condition for the stability of circular motion in the Schwarzschild field (Goldhammer, 1961). For $q \ll 1$, it follows that

$$r > 6m\left[1 - \frac{7q}{300}\right] \tag{5.20}$$

valid for both signs of q.

For q large and *negative*, it follows from equation (5.18) that stable motions are possible only for

$$r > \left(\frac{-6q}{55}\right)^{1/3}\left[1 + \frac{167}{396}\left(\frac{-6q}{55}\right)^{-1/3}\right]m \tag{5.21}$$

Finally, for q large and positive, we have from equation (5.18) that q must be less than a value given by

$$q = \frac{m}{g + h} \tag{5.22}$$

However, we recall that (5.22) also defines the maximum possible r for circular noncoplanar motions [see the discussion following equation (4.5)]. It is obvious from Figure 10 that these two limits coincide exactly for all $q > q_{\text{et}}$, that is, for all q for which circular noncoplanar motions exist. For $-\infty < q < q_{\text{et}}$, the minimum stable r is given by $r(q_1)$.

6. *Discussion*

Circular motions with constant angular velocity in planes parallel to the plane of symmetry of the quadrupole exist only for positive quadrupole moments. It was shown that for all r greater than an existence-threshold value $r_{\text{et}} = 2.4481m$, an ever-widening range of q values exists for which circular noncoplanar motions are possible. When $r = r_{\text{et}}$ this range shrinks to a point with the single allowed value $q = q_{\text{et}} = 2.2544$. The ranges $r < r_{\text{et}}$, and

$q < q_{et}$ are inaccessible as they require an orbital velocity exceeding the local velocity of light. Such restrictions do not occur in the Newtonian case. In both Newtonian mechanics and general relativity, the azimuth θ of the circular motions must exceed a minimum value. For the Newtonian case this is given by $\theta = \sin^{-1}(2/5)^{1/2}$. The corresponding restriction in general relativity involves an r-dependent condition which is everywhere more restrictive than the Newtonian one (Figure 2). The radius of the smallest noncoplanar circle is $R_{\min} = 2.37704m$, which is smaller than the smallest radial polar coordinate $r = r_{et}$ (Figure 3).

In the general relativistic case, stability thresholds for r and q were obtained having the values $r_{st} = 4.9061m$, and $q_{st} = 24.2333$. In addition, the relativistic stability condition on the azimuth is again more restrictive everywhere than the Newtonian one $\theta > \theta_{\min} = \sin^{-1}(8/15)^{1/2}$ (Figure 8).

A treatment of the existence and stability of circular motions in the plane of symmetry showed the existence of a stability region (Figure 10) which, unlike the existence region (Figure 4), widened continuously from a point, thereby defining stability thresholds on r and q. Remarkably enough, the stability-threshold values of r and q for motions in the plane of symmetry are precisely the existence-threshold values $r = r_{et}$ and $q = q_{et}$ obtained for circular noncoplanar motion. For $q = 0$, the existence and stability conditions reduce to $r > 3m$ and $r > 6m$, respectively, both of which are in agreement with the Schwarzschild results (Figure 4 and 10).

In both Newtonian and general relativistic cases, stable orbits of smaller radius are possible outside the plane of symmetry than within. In fact, in both theories, orbits outside the plane are stable up to a maximum value of the radius reached at the plane of symmetry, which coincides exactly with the minimum radius for stable circular orbits within that plane.

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